LMU-ASC 73/10

On possible Chern Classes of stable Bundles on Calabi-Yau threefolds

Björn Andreas[1$] and Gottfried Curio[2£]

[$]*Institut für Mathematik*
Humboldt-Universität zu Berlin
Rudower Chaussee 25, 12489 Berlin, Germany

[£]*Arnold-Sommerfeld-Center for Theoretical Physics*
Department für Physik, Ludwig-Maximilians-Universität München
Theresienstr. 37, 80333 München, Germany

Abstract

Supersymmetric heterotic string models, built from a Calabi-Yau threefold X endowed with a stable vector bundle V, usually lead to an anomaly mismatch between $c_2(V)$ and $c_2(X)$; this leads to the question whether the difference can be realized by a further bundle in the hidden sector. In math.AG/0604597 a conjecture is stated which gives sufficient conditions on cohomology classes on X to be realized as the Chern classes of a stable reflexive sheaf V; a weak version of this conjecture predicts the existence of such a V if $c_2(V)$ is of a certain form. In this note we prove that on elliptically fibered X infinitely many cohomology classes $c \in H^4(X, \mathbf{Z})$ exist which are of this form and for each of them a stable $SU(n)$ vector bundle with $c = c_2(V)$ exists.

[1]andreas@math.hu-berlin.de

[2]gottfried.curio@physik.uni-muenchen.de; supported by DFG grant CU 191/1-1

1 Introduction

To get $N = 1$ heterotic string models in four dimensions one compactifies the tendimensional heterotic string on a Calabi-Yau threefold X which is furthermore endowed with a polystable holomorphic vector bundle V'. Usually one takes $V' = (V, V_{hid})$ with V a stable bundle considered to be embedded in (the visible) E_8 (V_{hid} plays the corresponding role for the second hidden E_8); the commutator of V gives the unbroken gauge group in four dimensions.

The most important invariants of V are its Chern classes $c_i(V)$, $i = 0, 1, 2, 3$. We consider in this note bundles with $c_0(V) = rk(V) = n$ and $c_1(V) = 0$; more specifically we will consider $SU(n)$ bundles. The net number of generations of chiral particle multiplets in the fourdimensional effective theory is given by $N_{gen}(V) = c_3(V)/2$. On the other hand the second Chern class is important to assure anomaly freedom of the construction: this is encoded in the integrability condition for the existence of a solution to the anomaly cancellation equation

$$c_2(X) \quad = \quad c_2(V) + W. \tag{1.1}$$

Here W, as it stands, has just the meaning to indicate a possible mismatch for a certain bundle V; it can be understood either as the cohomology class of (the compact part of the world-volume of) a fivebrane, or as second Chern class of a further stable bundle V_{hid} in the hidden sector. Furthermore in the first case the class of W has to be effective for supersymmetry to be preserved.

Often one will argue just from the data provided by the Chern classes, say to secure a certain phenomenologically favored generation number, and so has to make sure that a corresponding $SU(n)$ bundle with suitably prescribed Chern class $c_3(V)$ actually exists. On the other hand, following the route to solve (1.1) described, one has the same problem for $c_2(V_{hid}) = W := c_2(X) - c_2(V_{vis})$ concerning the hidden bundle.

In [4] it has been shown that whenever the topological constraint can be satisfied with $W = 0$ then X and V can be deformed to a solution of the anomaly equation even already on the level of differential forms (a solution to the system involving the three-form field-strength H, investigated first in [5], exists).

This leads to the general question to give sufficient conditions for the existence of stable bundles with prescribed Chern classes $c_2(V)$ and $c_3(V)$. Concerning this the following conjecture is put forward in [1] by Douglas, Reinbacher and Yau (DRY) (actually we use the particular case of the conjecture with $c_1(V) = 0$).

DRY-Conjecture. *On a Calabi-Yau threefold X of $\pi_1(X) = 0$ a stable reflexive sheaf V of rank n and $c_1(V) = 0$ with prescribed Chern classes $c_2(V)$ and $c_3(V)$ will exist if, for an ample class $H \in H^2(X, \mathbf{R})$, these can be written as (where $C := 16\sqrt{2}/3$)*

$$c_2(V) = n\left(H^2 + \frac{c_2(X)}{24}\right) \tag{1.2}$$

$$c_3(V) < C\, nH^3. \tag{1.3}$$

Note that the conjecture just predicts the existence of a stable reflexive sheaf; in our examples below V will be a vector bundle.

We will also formulate a weaker version of the conjecture, which is implied by the proper (strong) form and concentrates on the existence of V given that just its (potential) $c_2(V)$ fulfills the relevant condition. To refer more easily to the notions involved, we make first the following definitions. We restrict to the case of V being a vector bundle. We will consider rank n bundles of $c_1(V) = 0$ and treat actually the case of $SU(n)$ vector bundles.

Definition. Let X be a Calabi-Yau threefold of $\pi_1(X) = 0$ and $c \in H^4(X, \mathbf{Z})$,
i) c is called a *Chern class* if a stable $SU(n)$ vector bundle V on X exists with $c = c_2(V)$
ii) c is called a *DRY class* if an ample class $H \in H^2(X, \mathbf{R})$ exists (and an integer n) with

$$c_2(V) = n\left(H^2 + \frac{c_2(X)}{24}\right). \tag{1.4}$$

With these definitions we can now state the weak DRY conjecture, in the framework as we will use it, as follows:

Weak DRY-Conjecture. *On a Calabi-Yau threefold X of $\pi_1(X) = 0$ every DRY class $c \in H^4(X, \mathbf{Z})$ is a Chern class.*

Here it is understood that the integer n occurring in the two definitions is the same.

The paper has three parts. In *section 2* we give criteria for a class to be a DRY class. In *section 3* we present some bundle constructions and show that their $c_2(V)$ fulfill these criteria for infinitely many V. In *section 4* we give an application in a physical set-up.

2 DRY classes on elliptic Calabi-Yau threefolds

To test these conjectures we choose X to be elliptically fibered over the base surface B with section $\sigma : B \longrightarrow X$ (we will also denote by σ the embedded subvariety $\sigma(B) \subset X$ and its cohomology class in $H^2(X, \mathbf{Z})$), a case particularly well studied. The typical examples for B are rational surfaces like a Hirzebruch surface $\mathbf{F_k}$ (where we consider

the following cases $k=0,1,2$ as only for these bases exists a smooth elliptic X with Weierstrass model), a del Pezzo surface $\mathbf{dP_k}$ ($k=0,\dots,8$) or the Enriques surface (or suitable blow-ups of these cases). We will consider specifically bases B for which $c_1 := c_1(B)$ is ample. This excludes in particular the Enriques surface and the Hirzebruch surface[3] $\mathbf{F_2}$. (The classes c_1^2 and $c_2 := c_2(B)$ will be considered as (integral) numbers.)

On the elliptic Calabi-Yau space X one has according to the general decomposition $H^4(X,\mathbf{Z}) \cong H^2(X,\mathbf{Z})\sigma \oplus H^4(B,\mathbf{Z})$ the decompositions (with $\phi,\rho \in H^2(X,\mathbf{Z})$)

$$c_2(V) = \phi\sigma + \omega \tag{2.1}$$

$$c_2(X) = 12c_1\sigma + c_2 + 11c_1^2 \tag{2.2}$$

where ω is understood as an integral number (pullbacks from B to X will be usually suppressed).

One now solves for $H = a\sigma + \rho$ (using the decomposition $H^2(X,\mathbf{Z}) \cong \mathbf{Z}\sigma \oplus H^2(B,\mathbf{Z})$), given an arbitrary but fixed class $c = \phi\sigma + \omega \in H^4(X,\mathbf{Z})$, and has then to check that H is ample. The conditions for H to be ample are (cf. appendix)

$$H \quad \text{ample} \qquad \Longleftrightarrow \qquad a>0, \quad \rho - ac_1 \;\; \text{ample}. \tag{2.3}$$

Inserting H into equ. (1.4) one gets the following relations

$$\phi = 2an\rho + n\Big(\frac{1}{2} - a^2\Big)c_1 \tag{2.4}$$

$$\omega = \rho^2 + \frac{5}{12}c_1^2 + \frac{1}{2} \tag{2.5}$$

(note that $\sigma^2 = -c_1\sigma$, cf. [2]) where in (2.5) Noethers relation $c_2 + c_1^2 = 12$ for the rational surface B has been used. This implies

$$\rho = \frac{1}{2an}\Big(\phi - n(\frac{1}{2} - a^2)c_1\Big) \tag{2.6}$$

As we assumed that c_1 is ample one finds that the condition that the class

$$\rho - ac_1 = \frac{1}{2an}\Big(\phi - n(\frac{1}{2} + a^2)c_1\Big) \tag{2.7}$$

is ample leads also to an upper bound on the positive real number a^2

$$0 \; < \; a^2 \; < \; b \tag{2.8}$$

This bound will be specified in an example below explicitly. Furthermore equ. (2.7) shows that $\phi - \frac{n}{2}c_1$ must necessarily be ample.

[3] as $c_1 \cdot b = (2b + 4f)\cdot b = 0$, using here the notations from footn. 6

Having solved by now (2.4) in terms of ρ one now has to solve the following equation

$$\omega = \frac{1}{4a^2n^2}\left(\phi - n(\frac{1}{2} - a^2)c_1\right)^2 + \frac{5}{12}c_1^2 + \frac{1}{2} \tag{2.9}$$

in terms of a. Actually the only non-trivial point will be that a is real and satisfies (2.8). Concretely one gets a quadratic equation $a^4 + pa^2 + q = 0$ in a^2 with

$$p = -\frac{4}{c_1^2}\left(\omega - r\right) \tag{2.10}$$

$$q = \frac{4}{(c_1^2)^2}s^2 \tag{2.11}$$

where we used the abbreviations

$$r := \frac{1}{2n}\phi c_1 + \frac{1}{6}c_1^2 + \frac{1}{2} \tag{2.12}$$

$$s := \frac{1}{2n}\sqrt{c_1^2}\sqrt{\left(\phi - \frac{n}{2}c_1\right)^2} \tag{2.13}$$

Now one has three conditions which have to be satisfied by at least one solution a_*^2 of this equation, namely[4]

$$i) \qquad a_*^2 \in \mathbf{R} \iff p^2 \geq 4q \iff (\omega - r)^2 \geq s^2 \tag{2.14}$$

$$ii) \qquad a_*^2 > 0 \iff -p > 0 \iff \omega > r \tag{2.15}$$

$$iii) \qquad a_*^2 \leq b \iff \begin{cases} -p < b + \frac{q}{b} & \text{for } +\surd \text{ and } b \geq -\frac{p}{2} \\ \text{arbitrary} & \text{for } -\surd \text{ and } b \geq -\frac{p}{2} \\ -p > b + \frac{q}{b} & \text{for } -\surd \text{ and } b < -\frac{p}{2} \end{cases} \tag{2.16}$$

Concerning ii) note that necessarily $q > 0$, cf. the remark after (2.8); furthermore the evaluation of the condition is independent of the question which sign for the square root is taken. Note that in iii) the case where $+\surd$ is taken and $b < -\frac{p}{2}$ is excluded.

Concerning condition iii) note that for $b \geq -p/2$ one gets no further restriction and has to pose *in total* just the first two conditions, i.e. $\omega \geq r + s$. By contrast for $b < -p/2$ the condition $-p > b + \frac{q}{b}$, or equivalently $\omega > \omega_0(\phi; b) := r + \frac{c_1^2}{4}(b + \frac{q}{b})$, implies i) and ii).

As p (and thus ω itself) occurs in the domain restrictions on b one has to rewrite these conditions slightly. Let us consider first the regime $b < -\frac{p}{2}$ which means explicitly $\omega > r + \frac{b}{2}c_1^2$. Now one has to distuinguish again two cases: the ensuing condition $\omega > \omega_0(\phi; b)$ makes sense as an (additional) condition (which would be to be required besides the domain restriction $\omega > r + \frac{b}{2}c_1^2$) only if $r + \frac{b}{2}c_1^2 < \omega_0(\phi; b)$, i.e. for $b < \sqrt{q}$; on the other hand for $b \geq \sqrt{q}$ one just has to demand $\omega > r + \frac{b}{2}c_1^2$.

[4]here it is understood that ii) assumes i) satisfied and iii) assumes i) and ii) satisfied

In the second regime $b \geq -p/2$, or equivalently $\omega \leq r + \frac{b}{2}c_1^2$, one has the condition $\omega \geq r+s$ (note that these two conditions are compatible just for $s \leq \frac{b}{2}c_1^2$, i.e. $b \geq \sqrt{q}$). Thus in total one can make an ω-independent regime distinction for b according to $b < \sqrt{q}$ (with the demand $\omega > \omega_0$) or $b \geq \sqrt{q}$ (where one should have either $\omega > r + \frac{b}{2}c_1^2$ or $\omega \leq r + \frac{b}{2}c_1^2$ but in that latter case one has to demand $\omega \geq r+s$; but, as $r+s \leq r + \frac{b}{2}c_1^2$ in the present b-regime, one just has to demand that $\omega \geq r+s$).

Note in this connection that, for ϕ held fixed, $\omega_0(\phi; b)$ becomes large for small and large b and the intermediate minimum is achieved at $b_{min} = \sqrt{q}$. As the condition $\omega > \omega_0(\phi; b)$ is relevant only for $b < \sqrt{q}$ whereas for $b > \sqrt{q}$ one gets the condition $\omega > r+s$ and as one has $\omega_0(\phi, b_{min}) = r+s$ there is a smooth transition in the conditions; and furthermore, all in all, $\omega \geq r+s$ is a necessary condition for a class to be a DRY class.

Therefore we get the following theorem

Theorem on DRY classes. *For a class $c = \phi\sigma + \omega \in H^4(X, \mathbf{Z})$ to be a DRY class one has the following conditions (where b is some $b \in \mathbf{R}^{>0}$ and $\omega \in H^4(B, \mathbf{Z}) \cong \mathbf{Z}$):*

a) necessary and sufficient: $\phi - n(\frac{1}{2}+b)c_1$ is ample and $\begin{cases} \omega \geq r+s & \text{for } b \geq \sqrt{q} \\ \omega > \omega_0(\phi; b) & \text{for } b < \sqrt{q} \end{cases}$

b) sufficient: $\phi - \frac{n}{2}c_1$ is ample and ω sufficiently large

c) necessary: $\phi - \frac{n}{2}c_1$ is ample and $\omega \geq r+s$.

Here part b) follows immediately from a) as the ample cone of B is an open set. So in particular the condition on ω can be fulfilled in any bundle construction which contains a (discrete) parameter μ in ω such that ω can become arbitrarily large if μ runs in its range of values (this strategy will be used for spectral and extension bundles).

Let us discuss further the conditions on ϕ and ω given in the theorem for $c = \phi\sigma + \omega$ to be a DRY class. As the notion of c being a DRY class does not involve any b one should compare these conditions for different b. One then realises that as b becomes larger the condition on ϕ becomes stronger and stronger; on the other hand as b increases from 0 to $\sqrt{q}$ (for a fixed ϕ) the condition on ω becomes weaker first, and then, from $\sqrt{q}$ on, remains unchanged. From this consideration one learns that it is enough to use b's in the interval $0 < b \leq \sqrt{q}$ as test parameters. That is the set of DRY classes is the union of allowed ranges of ϕ and ω for all these b.

Remark: Note that, although for b_{min} the condition on ω is as weak as possible, $\phi - n(\frac{1}{2} + b_{min})c_1$ might not be ample (cf. the example of the tangent bundle given below); nevertheless the ampleness condition on ϕ might be satisifed for a b where the bound $\omega \geq \omega_0(\phi; b)$ turns out to be more stringent (cf. again the example).[5]

[5]Note also the following property of $b_{min}(\phi)$: the zero class lies in the boundary of the ample cone;

3 Examples for the DRY-Conjecture

In this section we give examples of cohomology classes which are of DRY-form and appear as second Chern classes of stable $SU(n)$ vector bundles.

3.1 The tangent bundle

Let us see whether the cohomology class given by the second Chern class of the tangent bundle TX is detected by the weak DRY-Conjecture as a Chern class. For this we apply the Theorem above to see whether $c_2(X)$ is a DRY class. The minimum of $\omega_0(\phi; b)$ is taken at $b_{min} = 7/2$ but one finds $0 < b < 7/2$ as the allowed range for b (c_1 was assumed ample); so although $\omega_{TX} = 10c_1^2 + 12 \geq \omega_0(12c_1, b_{min}) = \frac{47}{12}c_1^2 + \frac{1}{2}$ is fulfilled one has to take another b which makes the bound $\omega \geq \omega_0(12c_1; b)$ more stringent; but $b = 3$, say, where the ω_0 becomes $(\frac{49}{48} + 2 + \frac{11}{12})c_1^2 + \frac{1}{2}$, will do. So $c_2(X)$ is a DRY class and the weak DRY-conjecture is fulfilled; as $c_3(X) = -60c_1^2$ is negative actually even the (proper) DRY-conjecture is true.

3.2 Spectral bundles

In case of spectral cover bundles [2] one has the following expression for ω

$$\omega = (\lambda^2 - \frac{1}{4})\frac{n}{2}\phi(\phi - nc_1) - \frac{n^3 - n}{24}c_1^2 \tag{3.1}$$

Here ϕ is an effective class in B with $\phi - nc_1$ also effective and λ is a half-integer satisfying the following conditions: λ is strictly half-integral for n being odd; for n even an integral λ requires $\phi \equiv c_1 \pmod 2$ while a strictly half-integral λ requires c_1 even. (In addition one has to assume that the linear system $|\phi|$ is base point free[6].)

Often one assumes, as we will do here, that $\phi - nc_1$ is not only effective but even ample in B. Then equ. (2.7) shows that we can take $b = 1/2$ as upper bound on a.

One has now to check whether the three conditions on a^2 given above can be fulfilled. According to part b) of the theorem in section 2 one learns that this is the case as ω increases to arbitrarily large values when the parameter λ is increasing.

so, if the condition on ϕ is considered for this limiting case, one finds $\phi - \frac{n}{2}c_1 = bnc_1$ such that (ϕ is proportional to c_1 and) $b = b_{min}(\phi)$.

[6] a base point is a point common to all members of the system $|\phi|$ of effective divisors which are linearly equivalent to the divisor ϕ (note that on B the cohomology class ϕ specifies uniquely a divisor class); on B a Hirzebruch surface $\mathbf{F_k}$ with base $\mathbf{P^1}$ b and fibre $\mathbf{P^1}$ f this amounts to $\phi \cdot b \geq 0$

Theorem. *i) On X an elliptic Calabi-Yau threefold the class $c_2(V) = c = \phi\sigma + \omega$ for V a spectral bundle (of discrete bundle parameters $\eta \in H^2(B, \mathbf{Z})$ and $\lambda \in \frac{1}{2}\mathbf{Z}$) satisfies the assumptions of the weak DRY-Conjecture on c for all but finitely many values of the parameter λ.*
ii) For the infinitely many classes $c \in H^4(X, \mathbf{Z})$ described in i) the weak *DRY-Conjecture is true.*
*iii) For the classes in ii) with negative λ the (*proper*) DRY-Conjecture is true.*

Here part ii) follows of course just from reversing the perspective: whereas in part i) one started from a given spectral bundle V and found a condition (λ^2 sufficiently large) that its $c_2(V)$ fulfills the assumption of the weak DRY-Conjecture, one then turns around the perspective in part ii), where one has trivially confirmed the existence of a stable bundle for a $c = c_2(V)$ which satisfies the assumptions of the weak DRY-Conjecture.

Part iii) follows as $c_3(V) = 2\lambda\phi(\phi - nc_1)$ is negative for λ negative as $\phi \neq 0$ is effective and $\phi - nc_1$ was assumed ample, so $\phi(\phi - nc_1)$ is positive (this argument underlies of course already part ii) as well).

3.3 Extension bundles

Stable vector bundles built as an extension of given stable bundles have been constructed on elliptic Calabi-Yau threefolds in [3]. Let E be a rank r H_B-stable vector bundle on the base B of the Calabi-Yau space with Chern classes $c_1(E) = 0$ and $c_2(E) = k$. The pullback bundle $\pi^* E$ is then shown to be stable on X with respect to the ample class $J = z\sigma + H_B$ where $H_B = hc_1$ (with $h \in \mathbf{R}^{>0}$) [3]. The bundle extension

$$0 \to \pi^* E \otimes \mathcal{O}_X(-D) \to V \to \mathcal{O}_X(rD) \to 0 \tag{3.2}$$

with $D = x\sigma + \alpha$ defines a stable rank $n = r + 1$ vector bundle if the numerical condition equ. (3.4) is satisfied. We consider here the case $x = -1$ for simplicity. For this bundle $c = \phi\sigma + \omega$ is given by

$$\phi = (n-1)\frac{n}{2}(2\alpha + c_1), \qquad \omega = k - (n-1)\frac{n}{2}\alpha^2 \tag{3.3}$$

As in the spectral case one now has to check whether the three conditions on a^2 given in section 2 can be fulfilled. This is the case according to part b) of the theorem in section 2 if α is chosen such that $2(n-1)\alpha + (n-2)c_1$ is ample and k is chosen sufficiently large. Note that this is in agreement with the condition that the extension can be chosen nonsplit if

$$\frac{n-1}{2}\Big[n^2\Big(\alpha(\alpha + c_1) + \frac{c_1^2}{3}\Big) - c_1(2\alpha + \frac{c_1}{3}) + 1\Big] - k < 0 \tag{3.4}$$

As above in the spectral bundle case we get here the following result.

Theorem. *i) On X an elliptic Calabi-Yau threefold the class $c_2(V) = c = \phi\sigma + \omega$ for V an extension bundle (of discrete bundle parameters $\alpha \in H^2(B, \mathbf{Z})$ and $k \in \mathbf{Z}$) satisfies the assumptions of the weak DRY-Conjecture on c for all but finitely many values of the parameter k.*
ii) For the infinitely many classes $c \in H^4(X, \mathbf{Z})$ described in i) the weak *DRY-Conjecture is true.*
*iii) For infinitely many classes $c \in H^4(X, \mathbf{Z})$ the (*proper*) DRY-Conjecture is true.*

As above section 3.2, part ii) follows from reversing the perspective: whereas in part i) one started from a given extension bundle V and found a condition (k sufficiently large) that its $c_2(V)$ fullfills the assumption of the weak DRY-Conjecture, one then turns around the perspective in part ii), where one has now trivially confirmed the existence of a stable bundle for a $c = c_2(V)$ which satisfies the assumptions of the weak DRY-Conjecture.

Part iii) follows as $c_3(V) = -\frac{(n-1)(n-2)}{3}(c_1^2 + 3\alpha(\alpha + c_1)) - 2k < 0$ for k sufficiently large.

4 Application

Let us finally come back to the motivating question from the introduction. We will take a stable bundle in the visible sector V_{vis} of the heterotic string and want to supplement this by a stable bundle V_{hid} of rank n_h such that the anomaly condition $c_2(V_{vis}) + c_2(V_{hid}) = c_2(X)$ is satisfied. To assure the existence of V_{hid} we will assume the weak DRY conjecture. So, concretely we will check whether $c := c_2(X) - c_2(V_{vis})$ is a DRY class.

Let us take $V_{vis} = \pi^* E$ where E on B is a bundle with $c_2(E) = k$, stable with respect to the ample class H_B on B. Thus in this case we have

$$\phi = 12c_1, \quad \omega = 10c_1^2 + 12 - k \tag{4.1}$$

and furthermore one gets the explicit expression for the bound

$$\omega_0 = \Big[\frac{6}{n_h} + \frac{1}{6} + \frac{b}{4} + \frac{(12 - \frac{n_h}{2})^2}{4bn_h^2}\Big]c_1^2 + \frac{1}{2}. \tag{4.2}$$

We will use part b) of the theorem of section 2. We get $12 - n_h(\frac{1}{2} + b) > 0$ from the ampleness condition on ϕ and $\omega \geq \omega_0$ as further condition; here we assume that we are in the regime $b < \sqrt{q} = \frac{12 - \frac{n_h}{2}}{n_h}$. Note further that the DRY conjecture does not specify a polarization with respect to which V_{hid} will be stable; so V_{vis} should be stable with respect

to an arbitrary ample class; this is true in our case $V_{vis} = \pi^* E$ on $B = \mathbf{P^2}$ according to Lemma 5.1 of [3].

Thus for example, for $n_h = 4$ and $b = \frac{1}{2}$ one finds that c is for $k \leq 11$ a DRY class.

B. A. thanks SFB 647 for support. G. C. thanks the DFG for support in the project CU 191/1-1 and SFB 647 and the FU Berlin for hospitality.

A Ample classes on elliptic Calabi-Yau threefolds

Let $H = a\sigma + \rho \in H^2(X, \mathbf{R}) \cong \mathbf{R}\sigma + H^2(B, \mathbf{R})$ be a class on the elliptic Calabi-Yau threefold X. Then one has *if c_1 is ample*

$$H \text{ ample} \iff a > 0,\ \rho - ac_1 \text{ ample.} \tag{A.1}$$

Consider first the "$\Longrightarrow$" direction: one has $a = H \cdot F > 0$ according to the Nakai-Moishezon criterion that H is ample just if $H^3 > 0, H^2 \cdot S > 0, H \cdot C > 0$ for all irrducible surfaces S and irreducible curves C in X; here this is applied to the fibre F. Furthermore, if c is an irreducible curve in B one has $(\rho - ac_1) \cdot c = H \cdot c\sigma > 0$; and one also has $(\rho - ac_1)^2 = H^2 \cdot \sigma > 0$, such that by the same criterion, applied now on B, indeed the class $\rho - ac_1$ is ample.

Consider now the "$\Longleftarrow$" direction: the class of an irreducible curve C in X is built from the class F and non-negative linear combinations of classes of the form $c\sigma$, where c is now the class of an irreducible curve in B; therefore, turning the previous arguments around, one ends up indeed with $H \cdot C > 0$. The classes of irreducible surfaces are in a similar way built from σ and the $\pi^* c$; for $H^2 \cdot \sigma$ one can again turn around the previous argument; this is not so however for $H^2 \cdot \pi^* c = ac(2\rho - ac_1)$; in this case we adopt the additional assumption that c_1 is ample, which implies that ρ, and therefore $2\rho - ac_1$ too, is also ample to get the required conclusion. Similarly one concludes for $H^3 = a[\rho^2 + (\rho - ac_1)(2\rho - ac_1)]$.

B Examples of one-parameter Calabi-Yau spaces

Although we treat in the main body of the paper the case of elliptic Calabi-Yau spaces X let us briefly comment here on the simpler case where X is a one-parameter space, i.e., $h^{1,1}(X) = 1$.

In this case one has the representations (with $k, t \in \mathbf{Z}$)

$$c = kJ^2 \tag{B.1}$$

$$c_2(X) = tJ^2 \tag{B.2}$$

where J is a generating element of $H^2(X, \mathbf{Z})$; for the ample class H one has $H = hJ$ with $h \in \mathbf{R}^{>0}$.

The condition for a class c to have DRY form becomes here

$$k = n\Big(h^2 + \frac{t}{24}\Big) \tag{B.3}$$

This amounts to the condition

$$k > n\frac{t}{24} \tag{B.4}$$

whereas the necessary Bogomolov inequality $c \cdot J > 0$ gives just $k > 0$ (for example on the quintic one gets the stronger condition $k > \frac{5}{12}n$). Note that the second Chern class of the tangent bundle always has DRY-form; thus for this cohomology class the weak DRY-conjecture is satisfied, and for negative Euler number even the (proper) DRY-conjecture.

Some examples are provided by the complete intersection spaces $\mathbf{P^4}(5)$, $\mathbf{P}^5(2,4)$, $\mathbf{P^5}(3,3)$, $\mathbf{P^6}(2,2,3)$, $\mathbf{P^7}(2,2,2,2)$ with $t = 10, 7, 6, 5, 4$ and Euler numbers -200, -176, -144, -144, -128. (similarly one can discuss the one parameter cases $\mathbf{P_{2,1,1,1,1}}(6)$, $\mathbf{P_{4,1,1,1,1}}(8)$, $\mathbf{P_{5,2,1,1,1}}(10)$).

On the quintic one has some further bundles, occurring in the list in [6], with $c_2(V) = c_2(X)$ with some of them (the first five examples) shown to be stable in [7], which have the same t as TX and also negative $c_3(V)$; thus these provide further examples of the weak DRY-conjecture and actually even of the (proper) DRY-Conjecture.

Physically one has to demand in addition anomaly cancellation. Thus one gets then in total the condition

$$\frac{n}{24}t < k \le t \tag{B.5}$$

(note that one has here $k_{hid} > 0$ for a potential hidden bundle from the Bogomolov inequality).

For the generation number one gets, in the framework of the assumptions of the DRY conjecture, the bound

$$N_{gen} < C\frac{n}{2}\Big(\frac{k}{n} - \frac{t}{24}\Big)^{3/2}. \tag{B.6}$$

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LMU-ASC 73/10

On possible Chern Classes of stable Bundles on Calabi-Yau threefolds

Björn Andreas[1$] and Gottfried Curio[2£]

[$] *Institut für Mathematik*
Humboldt-Universität zu Berlin
Rudower Chaussee 25, 12489 Berlin, Germany

[£] *Arnold-Sommerfeld-Center for Theoretical Physics*
Department für Physik, Ludwig-Maximilians-Universität München
Theresienstr. 37, 80333 München, Germany

Abstract

Supersymmetric heterotic string models, built from a Calabi-Yau threefold X endowed with a stable vector bundle V, usually lead to an anomaly mismatch between $c_2(V)$ and $c_2(X)$; this leads to the question whether the difference can be realized by a further bundle in the hidden sector. In math.AG/0604597 a conjecture is stated which gives sufficient conditions on cohomology classes on X to be realized as the Chern classes of a stable reflexive sheaf V; a weak version of this conjecture predicts the existence of such a V if $c_2(V)$ is of a certain form. In this note we prove that on elliptically fibered X infinitely many cohomology classes $c \in H^4(X, \mathbf{Z})$ exist which are of this form and for each of them a stable $SU(n)$ vector bundle with $c = c_2(V)$ exists.

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[1] andreas@math.hu-berlin.de

[2] gottfried.curio@physik.uni-muenchen.de;
 supported by DFG grant CU 191/1-1

1 Introduction

To get $N = 1$ heterotic string models in four dimensions one compactifies the tendimensional heterotic string on a Calabi-Yau threefold X which is furthermore endowed with a polystable holomorphic vector bundle V'. Usually one takes $V' = (V, V_{hid})$ with V a stable bundle considered to be embedded in (the visible) E_8 (V_{hid} plays the corresponding role for the second hidden E_8); the commutator of V gives the unbroken gauge group in four dimensions.

The most important invariants of V are its Chern classes $c_i(V)$, $i = 0, 1, 2, 3$. We consider in this note bundles with $c_0(V) = rk(V) = n$ and $c_1(V) = 0$; more specifically we will consider $SU(n)$ bundles. The net number of generations of chiral particle multiplets in the fourdimensional effective theory is given by $N_{gen}(V) = c_3(V)/2$. On the other hand the second Chern class is important to assure anomaly freedom of the construction: this is encoded in the integrability condition for the existence of a solution to the anomaly cancellation equation

$$c_2(X) \quad = \quad c_2(V) + W. \tag{1.1}$$

Here W, as it stands, has just the meaning to indicate a possible mismatch for a certain bundle V; it can be understood either as the cohomology class of (the compact part of the world-volume of) a fivebrane, or as second Chern class of a further stable bundle V_{hid} in the hidden sector. Furthermore in the first case the class of W has to be effective for supersymmetry to be preserved.

Often one will argue just from the data provided by the Chern classes, say to secure a certain phenomenologically favored generation number, and so has to make sure that a corresponding $SU(n)$ bundle with suitably prescribed Chern class $c_3(V)$ actually exists. On the other hand, following the route to solve (1.1) described, one has the same problem for $c_2(V_{hid}) = W := c_2(X) - c_2(V_{vis})$ concerning the hidden bundle.

In [4] it has been shown that whenever the topological constraint can be satisfied with $W = 0$ then X and V can be deformed to a solution of the anomaly equation even already on the level of differential forms (a solution to the system involving the three-form field-strength H, investigated first in [5], exists).

This leads to the general question to give sufficient conditions for the existence of stable bundles with prescribed Chern classes $c_2(V)$ and $c_3(V)$. Concerning this the following conjecture is put forward in [1] by Douglas, Reinbacher and Yau (DRY) (actually we use the particular case of the conjecture with $c_1(V) = 0$).

DRY-Conjecture. *On a Calabi-Yau threefold X of $\pi_1(X)=0$ a stable reflexive sheaf V of rank n and $c_1(V)=0$ with prescribed Chern classes $c_2(V)$ and $c_3(V)$ will exist if, for an ample class $H \in H^2(X, \mathbf{R})$, these can be written as (where $C := 16\sqrt{2}/3$)*

$$c_2(V) \;=\; n\left(H^2 + \frac{c_2(X)}{24}\right) \tag{1.2}$$

$$c_3(V) \;<\; C\, nH^3. \tag{1.3}$$

Note that the conjecture just predicts the existence of a stable reflexive sheaf; in our examples below V will be a vector bundle.

We will also formulate a weaker version of the conjecture, which is implied by the proper (strong) form and concentrates on the existence of V given that just its (potential) $c_2(V)$ fulfills the relevant condition. To refer more easily to the notions involved, we make first the following definitions. We restrict to the case of V being a vector bundle. We will consider rank n bundles of $c_1(V)=0$ and treat actually the case of $SU(n)$ vector bundles.

Definition. Let X be a Calabi-Yau threefold of $\pi_1(X)=0$ and $c \in H^4(X, \mathbf{Z})$,
i) c is called a *Chern class* if a stable $SU(n)$ vector bundle V on X exists with $c = c_2(V)$
ii) c is called a *DRY class* if an ample class $H \in H^2(X, \mathbf{R})$ exists (and an integer n) with

$$c_2(V) \;=\; n\left(H^2 + \frac{c_2(X)}{24}\right). \tag{1.4}$$

With these definitions we can now state the weak DRY conjecture, in the framework as we will use it, as follows:

Weak DRY-Conjecture. *On a Calabi-Yau threefold X of $\pi_1(X)=0$ every DRY class $c \in H^4(X, \mathbf{Z})$ is a Chern class.*

Here it is understood that the integer n occurring in the two definitions is the same.

The paper has three parts. In *section 2* we give criteria for a class to be a DRY class. In *section 3* we present some bundle constructions and show that their $c_2(V)$ fulfill these criteria for infinitely many V.

2 DRY classes on elliptic Calabi-Yau threefolds

To test these conjectures we choose X to be elliptically fibered over the base surface B with section $\sigma : B \longrightarrow X$ (we will also denote by σ the embedded subvariety $\sigma(B) \subset X$ and its cohomology class in $H^2(X, \mathbf{Z})$), a case particularly well studied. The typical examples for B are rational surfaces like a Hirzebruch surface $\mathbf{F_k}$ (where we consider

the following cases $k = 0, 1, 2$ as only for these bases exists a smooth elliptic X with Weierstrass model), a del Pezzo surface $\mathbf{dP_k}$ ($k = 0, \ldots, 8$) or the Enriques surface (or suitable blow-ups of these cases). We will consider specifically bases B for which $c_1 := c_1(B)$ is ample. This excludes in particular the Enriques surface and the Hirzebruch surface[3] $\mathbf{F_2}$. (The classes c_1^2 and $c_2 := c_2(B)$ will be considered as (integral) numbers.)

On the elliptic Calabi-Yau space X one has according to the general decomposition $H^4(X, \mathbf{Z}) \cong H^2(X, \mathbf{Z})\sigma \oplus H^4(B, \mathbf{Z})$ the decompositions (with $\phi, \rho \in H^2(X, \mathbf{Z})$)

$$c_2(V) = \phi\sigma + \omega \tag{2.1}$$

$$c_2(X) = 12c_1\sigma + c_2 + 11c_1^2 \tag{2.2}$$

where ω is understood as an integral number (pullbacks from B to X will be usually suppressed).

One now solves for $H = a\sigma + \rho$ (using the decomposition $H^2(X, \mathbf{Z}) \cong \mathbf{Z}\sigma \oplus H^2(B, \mathbf{Z})$), given an arbitrary but fixed class $c = \phi\sigma + \omega \in H^4(X, \mathbf{Z})$, and has then to check that H is ample. The conditions for H to be ample are (cf. appendix)

$$H \quad \text{ample} \qquad \Longleftrightarrow \qquad a > 0, \quad \rho - ac_1 \ \text{ ample.} \tag{2.3}$$

Inserting H into equ. (1.4) one gets the following relations

$$\phi = 2an\rho + n\Big(\frac{1}{2} - a^2\Big)c_1 \tag{2.4}$$

$$\frac{1}{n}\omega = \rho^2 + \frac{5}{12}c_1^2 + \frac{1}{2} \tag{2.5}$$

(note that $\sigma^2 = -c_1\sigma$, cf. [2]) where in (2.5) Noethers relation $c_2 + c_1^2 = 12$ for the rational surface B has been used. This implies

$$\rho = \frac{1}{2an}\Big(\phi - n(\frac{1}{2} - a^2)c_1\Big) \tag{2.6}$$

As we assumed that c_1 is ample one finds that the condition that the class

$$\rho - ac_1 = \frac{1}{2an}\Big(\phi - n(\frac{1}{2} + a^2)c_1\Big) \tag{2.7}$$

is ample leads also to an upper bound on the positive real number a^2

$$0 < a^2 < b \tag{2.8}$$

This bound will be specified in an example below explicitly. Furthermore equ. (2.7) shows that $\phi - \frac{n}{2}c_1$ must necessarily be ample.

[3] as $c_1 \cdot b = (2b + 4f) \cdot b = 0$, using here the notations from footn. 6

Having solved by now (2.4) in terms of ρ one now has to solve the following equation

$$\frac{1}{n}\omega = \frac{1}{4a^2n^2}\Big(\phi - n(\frac{1}{2} - a^2)c_1\Big)^2 + \frac{5}{12}c_1^2 + \frac{1}{2} \tag{2.9}$$

in terms of a. Actually the only non-trivial point will be that a is real and satisfies (2.8). Concretely one gets a quadratic equation $a^4 + pa^2 + q = 0$ in a^2 with

$$p = -\frac{4}{c_1^2}\Big(\frac{1}{n}\omega - r\Big) \tag{2.10}$$

$$q = \frac{4}{(c_1^2)^2}s^2 \tag{2.11}$$

where we used the abbreviations

$$r := \frac{1}{2n}\phi c_1 + \frac{1}{6}c_1^2 + \frac{1}{2} \tag{2.12}$$

$$s := \frac{1}{2n}\sqrt{c_1^2}\sqrt{\Big(\phi - \frac{n}{2}c_1\Big)^2} \tag{2.13}$$

Now one has three conditions which have to be satisfied by at least one solution a_*^2 of this equation, namely[4]

$$i) \qquad a_*^2 \in \mathbf{R} \iff p^2 \geq 4q \iff \Big(\frac{1}{n}\omega - r\Big)^2 \geq s^2 \tag{2.14}$$

$$ii) \qquad a_*^2 > 0 \iff -p > 0 \iff \frac{1}{n}\omega > r \tag{2.15}$$

$$iii) \qquad a_*^2 \leq b \iff \begin{cases} -p < b + \frac{q}{b} & \text{for } +\surd \text{ and } b \geq -\frac{p}{2} \\ \text{arbitrary} & \text{for } -\surd \text{ and } b \geq -\frac{p}{2} \\ -p > b + \frac{q}{b} & \text{for } -\surd \text{ and } b < -\frac{p}{2} \end{cases} \tag{2.16}$$

Concerning ii) note that necessarily $q > 0$, cf. the remark after (2.8); furthermore the evaluation of the condition is independent of the question which sign for the square root is taken. Note that in iii) the case where $+\surd$ is taken and $b < -\frac{p}{2}$ is excluded.

Concerning condition iii) note that for $b \geq -p/2$ one gets no further restriction and has to pose *in total* just the first two conditions, i.e. $\frac{1}{n}\omega \geq r + s$. By contrast for $b < -p/2$ the condition $-p > b + \frac{q}{b}$, equivalently $\frac{1}{n}\omega > \omega_0(\phi; b) := r + \frac{c_1^2}{4}(b + \frac{q}{b})$, implies i) and ii).

As p (and thus ω) occurs in the domain restrictions on b one has to rewrite these conditions slightly. Consider first the regime $b < -\frac{p}{2}$, or explicitly $\frac{1}{n}\omega > r + \frac{b}{2}c_1^2$, and distuinguish two cases: the ensuing condition $\frac{1}{n}\omega > \omega_0(\phi; b)$ makes sense as an *additional* condition (required besides the domain restriction $\frac{1}{n}\omega > r + \frac{b}{2}c_1^2$) only if $r + \frac{b}{2}c_1^2 < \omega_0(\phi; b)$, i.e. for $b < \sqrt{q}$; on the other hand for $b \geq \sqrt{q}$ one just has to demand $\frac{1}{n}\omega > r + \frac{b}{2}c_1^2$.

[4]here it is understood that ii) assumes i) satisfied and iii) assumes i) and ii) satisfied

In the second regime $b \geq -p/2$, or equivalently $\frac{1}{n}\omega \leq r + \frac{b}{2}c_1^2$, one has the condition $\frac{1}{n}\omega \geq r + s$ (note that these two conditions are compatible just for $s \leq \frac{b}{2}c_1^2$, i.e. $b \geq \sqrt{q}$). Thus in total one can make an ω-independent regime distinction for b according to $b < \sqrt{q}$ (with the demand $\frac{1}{n}\omega > \omega_0$) or $b \geq \sqrt{q}$ (where one should have either $\frac{1}{n}\omega > r + \frac{b}{2}c_1^2$ or $\frac{1}{n}\omega \leq r + \frac{b}{2}c_1^2$ but in that latter case one has to demand $\frac{1}{n}\omega \geq r + s$; but, as $r + s \leq r + \frac{b}{2}c_1^2$ in the present b-regime, one just has to demand that $\frac{1}{n}\omega \geq r + s$).

Note in this connection that, for ϕ held fixed, $\omega_0(\phi; b)$ becomes large for small and large b and the intermediate minimum is achieved at $b_{min} = \sqrt{q}$. As the condition $\frac{1}{n}\omega > \omega_0(\phi; b)$ is relevant only for $b < \sqrt{q}$ whereas for $b > \sqrt{q}$ one gets the condition $\frac{1}{n}\omega > r + s$ and as one has $\omega_0(\phi, b_{min}) = r + s$ there is a smooth transition in the conditions; and, all in all, $\frac{1}{n}\omega \geq r + s$ is a necessary condition for a class to be a DRY class.

Therefore we get the following theorem

Theorem on DRY classes. *For a class $c = \phi\sigma + \omega \in H^4(X, \mathbf{Z})$ to be a DRY class one has the following conditions (where b is some $b \in \mathbf{R}^{>0}$, $b < \sqrt{q}$, and $\omega \in H^4(B, \mathbf{Z}) \cong \mathbf{Z}$):*
a) necessary and sufficient: $\phi - n(\frac{1}{2} + b)c_1$ is ample and $\frac{1}{n}\omega > \omega_0(\phi; b)$
b) sufficient: $\phi - \frac{n}{2}c_1$ is ample and ω sufficiently large
c) necessary: $\phi - \frac{n}{2}c_1$ is ample and $\frac{1}{n}\omega \geq r + s$.

Here part b) follows immediately from a) as the ample cone of B is an open set. So in particular the condition on ω can be fulfilled in any bundle construction which contains a (discrete) parameter μ in ω such that ω can become arbitrarily large if μ runs in its range of values (this strategy will be used for spectral and extension bundles).

Note that as the notion of c being a DRY class does not involve any b one should compare these conditions for different b. One then realises that as b becomes larger the condition on ϕ becomes stronger and stronger; on the other hand as b increases from 0 to $\sqrt{q}$ (for a fixed ϕ) the condition on ω becomes weaker. Note that, assuming that $\phi - \frac{\mathcal{N}}{2}c_1 = A + b\mathcal{N}c_1$ with an ample class A on B, one has $(\phi - \frac{\mathcal{N}}{2}c_1)^2 > b^2\mathcal{N}^2c_1^2$, which is $q > b^2$. So it is enough to use b's in the interval $0 < b < \sqrt{q}$ as test parameters. That is the set of DRY classes is the union of allowed ranges of ϕ and ω for all these b.

Remark: Note that, although for b_{min} the condition on ω is as weak as possible, $\phi - n(\frac{1}{2} + b_{min})c_1$ might not be ample (cf. the example of the tangent bundle given below); nevertheless the ampleness condition on ϕ might be satisifed for a b where the bound $\omega \geq \omega_0(\phi; b)$ turns out to be more stringent (cf. again the example).[5]

[5] Note also the following property of $b_{min}(\phi)$: the zero class lies in the boundary of the ample cone; so, if the condition on ϕ is considered for this limiting case, one finds $\phi - \frac{n}{2}c_1 = bnc_1$ such that (ϕ is proportional to c_1 and) $b = b_{min}(\phi)$.

3 Examples for the DRY-Conjecture

In this section we give examples of cohomology classes which are of DRY-form and appear as second Chern classes of stable $SU(n)$ vector bundles.

3.1 The tangent bundle

Let us see whether the cohomology class given by $c_2(TX)$ is detected by the weak DRY-Conjecture as a Chern class. For this we apply the Theorem above to see whether $c_2(X)$ is a DRY class. The minimum of $\omega_0(\phi; b)$ is assumed at $b_{min} = \sqrt{q} = 7/2$ but one finds $0 < b < 7/2$ as the allowed range for b (c_1 was assumed ample); furthermore one has $\frac{1}{3}\omega_{TX} = \frac{10}{3}c_1^2 + 4 \geq \omega_0(12c_1, b_{min}) = r + s = \frac{47}{12}c_1^2 + \frac{1}{2}$ only for $c_1^2 \leq 6$, but one has in any case to take a smaller b which makes the bound $\omega \geq \omega_0(12c_1; b)$ even more stringent. It suffices however to take b minimally smaller (which is also optimal for the bound $\frac{1}{n}\omega \geq \omega_0(12c_1, b)$ as $\omega_0(12c_1, b)$ becomes minimally greater for b becoming minimally smaller), such that one gets $c_1^2 < 6$ as precise condition for $c_2(TX)$ to be a DRY-class; i.e., in these cases the weak DRY-conjecture is fulfilled (as $c_3(X) = -60c_1^2$ is negative actually even the (proper) DRY-conjecture is true); by contrast the cases $c_1^2 \geq 6$ illustrate that being a DRY-class is only a sufficient condition for a class to be realised as Chern class of a stable bundle, but not a necessary one.

3.2 Spectral bundles

In case of spectral cover bundles [2] one has the following expression for ω

$$\omega \;=\; (\lambda^2 - \frac{1}{4})\frac{n}{2}\phi(\phi - nc_1) - \frac{n^3 - n}{24}c_1^2 \tag{3.1}$$

Here ϕ is an effective class in B with $\phi - nc_1$ also effective and λ is a half-integer satisfying the following conditions: λ is strictly half-integral for n being odd; for n even an integral λ requires $\phi \equiv c_1 \pmod 2$ while a strictly half-integral λ requires c_1 even. (In addition one has to assume that the linear system $|\phi|$ is base point free[6].)

Often one assumes, as we will do here, that $\phi - nc_1$ is not only effective but even ample in B. Then equ. (2.7) shows that we can take $b = 1/2$ as upper bound on a.

[6] a base point is a point common to all members of the system $|\phi|$ of effective divisors which are linearly equivalent to the divisor ϕ (note that on B the cohomology class ϕ specifies uniquely a divisor class); on B a Hirzebruch surface $\mathbf{F_k}$ with base $\mathbf{P^1}$ b and fibre $\mathbf{P^1}$ f this amounts to $\phi \cdot b \geq 0$

One has now to check whether the three conditions on a^2 given above can be fulfilled. According to part b) of the theorem in section 2 one learns that this is the case as ω increases to arbitrarily large values when the parameter λ is increasing.

Theorem. *i) On X an elliptic Calabi-Yau threefold the class $c_2(V) = c = \phi\sigma + \omega$ for V a spectral bundle (of discrete bundle parameters $\eta \in H^2(B, \mathbf{Z})$ and $\lambda \in \frac{1}{2}\mathbf{Z}$) satisfies the assumptions of the weak DRY-Conjecture on c for all but finitely many values of the parameter λ.*
ii) For the infinitely many classes $c \in H^4(X, \mathbf{Z})$ described in i) the weak *DRY-Conjecture is true.*
*iii) For the classes in ii) with negative λ the (*proper*) DRY-Conjecture is true.*

Here part ii) follows of course just from reversing the perspective: whereas in part i) one started from a given spectral bundle V and found a condition (λ^2 sufficiently large) that its $c_2(V)$ fulfills the assumption of the weak DRY-Conjecture, one then turns around the perspective in part ii), where one has trivially confirmed the existence of a stable bundle for a $c = c_2(V)$ which satisfies the assumptions of the weak DRY-Conjecture.

Part iii) follows as $c_3(V) = 2\lambda\phi(\phi - nc_1)$ is negative for λ negative as $\phi \neq 0$ is effective and $\phi - nc_1$ was assumed ample, so $\phi(\phi - nc_1)$ is positive (this argument underlies of course already part ii) as well).

3.3 Extension bundles

Stable vector bundles built as an extension of given stable bundles have been constructed on elliptic Calabi-Yau threefolds in [3]. Let E be a rank r H_B-stable vector bundle on the base B of the Calabi-Yau space with Chern classes $c_1(E) = 0$ and $c_2(E) = k$. The pullback bundle π^*E is then shown to be stable on X with respect to the ample class $J = z\sigma + H_B$ where $H_B = hc_1$ (with $h \in \mathbf{R}^{>0}$) [3]. The bundle extension

$$0 \to \pi^*E \otimes \mathcal{O}_X(-D) \to V \to \mathcal{O}_X(rD) \to 0 \tag{3.2}$$

with $D = x\sigma + \alpha$ defines a stable rank $n = r + 1$ vector bundle if the numerical condition equ. (3.4) is satisfied. We consider here the case $x = -1$ for simplicity. For this bundle $c = \phi\sigma + \omega$ is given by

$$\phi = (n-1)\frac{n}{2}(2\alpha + c_1), \qquad \omega = k - (n-1)\frac{n}{2}\alpha^2 \tag{3.3}$$

As in the spectral case one now has to check whether the three conditions on a^2 given in section 2 can be fulfilled. This is the case according to part b) of the theorem in section

2 if α is chosen such that $2(n-1)\alpha + (n-2)c_1$ is ample and k is chosen sufficiently large. Note that this is in agreement with the condition that the extension can be chosen nonsplit if

$$\frac{n-1}{2}\Big[n^2\Big(\alpha(\alpha+c_1)+\frac{c_1^2}{3}\Big) - c_1(2\alpha+\frac{c_1}{3})+1\Big] - k < 0 \tag{3.4}$$

As above in the spectral bundle case we get here the following result.

Theorem. *i) On X an elliptic Calabi-Yau threefold the class $c_2(V) = c = \phi\sigma + \omega$ for V an extension bundle (of discrete bundle parameters $\alpha \in H^2(B, \mathbf{Z})$ and $k \in \mathbf{Z}$) satisfies the assumptions of the weak DRY-Conjecture on c for all but finitely many values of the parameter k.*
ii) For the infinitely many classes $c \in H^4(X, \mathbf{Z})$ described in i) the weak *DRY-Conjecture is true.*
*iii) For infinitely many classes $c \in H^4(X, \mathbf{Z})$ the (*proper*) DRY-Conjecture is true.*

As above section 3.2, part ii) follows from reversing the perspective: whereas in part i) one started from a given extension bundle V and found a condition (k sufficiently large) that its $c_2(V)$ fullfills the assumption of the weak DRY-Conjecture, one then turns around the perspective in part ii), where one has now trivially confirmed the existence of a stable bundle for a $c = c_2(V)$ which satisfies the assumptions of the weak DRY-Conjecture.

Part iii) follows as $c_3(V) = -\frac{(n-1)(n-2)}{3}(c_1^2 + 3\alpha(\alpha + c_1)) - 2k < 0$ for k sufficiently large.

3.4 A further Example

Let us finally come back to the motivating question from the introduction. We will take a stable bundle in the visible sector V_{vis} of the heterotic string and want to supplement this by a stable bundle V_{hid} of rank n_h such that the anomaly condition $c_2(V_{vis}) + c_2(V_{hid}) = c_2(X)$ is satisfied. To assure the existence of V_{hid} we will assume the weak DRY conjecture. So, concretely we will check whether $c := c_2(X) - c_2(V_{vis})$ is a DRY class.

Let us take $V_{vis} = \pi^* E$ where E on B is a bundle with $c_2(E) = k$, stable with respect to the ample class H_B on B. Thus in this case we have

$$\phi = 12c_1, \quad \omega = 10c_1^2 + 12 - k \tag{3.5}$$

and furthermore one gets the explicit expression for the bound

$$\omega_0 = \Big[\frac{6}{n_h} + \frac{1}{6} + \frac{b}{4} + \frac{(12 - \frac{n_h}{2})^2}{4bn_h^2}\Big]c_1^2 + \frac{1}{2}. \tag{3.6}$$

Let us consider part a) of the theorem of section 2. We get $12 - n_h(\frac{1}{2} + b) > 0$ from the ampleness condition (so we are in the regime $b < \sqrt{q} = \frac{12 - \frac{n_h}{2}}{n_h}$) on ϕ and $\frac{1}{n_h}\omega \geq \omega_0$ as further condition. Note further that the DRY conjecture does not specify a polarization with respect to which V_{hid} will be stable; so, to get a polystable bundle in total, V_{vis} should be stable with respect to an arbitrary ample class; this is true in our case $V_{vis} = \pi^* E$ only for $B = \mathbf{P^2}$ (where $H^{1,1}(B)$ is onedimensional) according to Lemma 5.1 of [3]. This restriction is however in contradiction with the necessary condition $\frac{1}{n_h}\omega \geq r + s$ from which one finds $c_1^2 \leq \frac{12-k-n_h/2}{2-n_h/12}$. Thus, for this (rather special) example of V_{vis} one does not succeed in complementing (in the sense of satisfying the anomaly equation) V_{vis} by a hidden bundle. In many more relevant examples for V_{vis}, however, this strategy succeedes [8].

B. A. thanks SFB 647 for support. G. C. thanks the DFG for support in the project CU 191/1-1 and SFB 647 and the FU Berlin for hospitality.

A Ample classes on elliptic Calabi-Yau threefolds

Let $H = a\sigma + \rho \in H^2(X, \mathbf{R}) \cong \mathbf{R}\sigma + H^2(B, \mathbf{R})$ be a class on the elliptic Calabi-Yau threefold X. Then one has *if* c_1 *is ample*

$$H \text{ ample} \iff a > 0,\ \rho - ac_1 \text{ ample}. \tag{A.1}$$

Consider first the "$\Longrightarrow$" direction: one has $a = H \cdot F > 0$ according to the Nakai-Moishezon criterion that H is ample just if $H^3 > 0, H^2 \cdot S > 0, H \cdot C > 0$ for all irrducible surfaces S and irreducible curves C in X; here this is applied to the fibre F. Furthermore, if c is an irreducible curve in B one has $(\rho - ac_1) \cdot c = H \cdot c\sigma > 0$; and one also has $(\rho - ac_1)^2 = H^2 \cdot \sigma > 0$, such that by the same criterion, applied now on B, indeed the class $\rho - ac_1$ is ample.

Consider now the "$\Longleftarrow$" direction: the class of an irreducible curve C in X is built from the class F and non-negative linear combinations of classes of the form $c\sigma$, where c is now the class of an irreducible curve in B; therefore, turning the previous arguments around, one ends up indeed with $H \cdot C > 0$. The classes of irreducible surfaces are in a similar way built from σ and the $\pi^* c$; for $H^2 \cdot \sigma$ one can again turn around the previous argument; this is not so however for $H^2 \cdot \pi^* c = ac(2\rho - ac_1)$; in this case we adopt the additional assumption that c_1 is ample, which implies that ρ, and therefore $2\rho - ac_1$ too, is also ample to get the required conclusion. Similarly one concludes for $H^3 = a[\rho^2 + (\rho - ac_1)(2\rho - ac_1)]$.

B Examples of one-parameter Calabi-Yau spaces

Although we treat in the main body of the paper the case of elliptic Calabi-Yau spaces X let us briefly comment here on the simpler case where X is a one-parameter space, i.e., $h^{1,1}(X) = 1$.

In this case one has the representations (with $k, t \in \mathbf{Z}$)

$$c = kJ^2 \tag{B.1}$$

$$c_2(X) = tJ^2 \tag{B.2}$$

where J is a generating element of $H^2(X, \mathbf{Z})$; for the ample class H one has $H = hJ$ with $h \in \mathbf{R}^{>0}$.

The condition for a class c to have DRY form becomes here

$$k = n\Big(h^2 + \frac{t}{24}\Big) \tag{B.3}$$

This amounts to the condition

$$k > n\frac{t}{24} \tag{B.4}$$

whereas the necessary Bogomolov inequality $c \cdot J > 0$ gives just $k > 0$ (for example on the quintic one gets the stronger condition $k > \frac{5}{12}n$). Note that the second Chern class of the tangent bundle always has DRY-form; thus for this cohomology class the weak DRY-conjecture is satisfied, and for negative Euler number even the (proper) DRY-conjecture.

Some examples are provided by the complete intersection spaces $\mathbf{P^4}(5)$, $\mathbf{P}^5(2,4)$, $\mathbf{P^5}(3,3)$, $\mathbf{P^6}(2,2,3)$, $\mathbf{P^7}(2,2,2,2)$ with $t = 10, 7, 6, 5, 4$ and Euler numbers -200, -176, -144, -144, -128. (similarly one can discuss the one parameter cases $\mathbf{P_{2,1,1,1,1}}(6)$, $\mathbf{P_{4,1,1,1,1}}(8)$, $\mathbf{P_{5,2,1,1,1}}(10)$).

On the quintic one has some further bundles, occurring in the list in [6], with $c_2(V) = c_2(X)$ with some of them (the first five examples) shown to be stable in [7], which have the same t as TX and also negative $c_3(V)$; thus these provide further examples of the weak DRY-conjecture and actually even of the (proper) DRY-Conjecture.

Physically one has to demand in addition anomaly cancellation. Thus one gets then in total the condition

$$\frac{n}{24}t < k \leq t \tag{B.5}$$

(note that one has here $k_{hid} > 0$ for a potential hidden bundle from the Bogomolov inequality).

For the generation number one gets, in the framework of the assumptions of the DRY conjecture, the bound

$$N_{gen} < C\frac{n}{2}\Big(\frac{k}{n} - \frac{t}{24}\Big)^{3/2}. \tag{B.6}$$

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